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Engineering Notes

Analysis of Transition Stability for Morphing Aircraft

T. M. Seigler*
University of Kentucky, Lexington, Kentucky 40506-0503
and
D. A. Neal†
Barron Associates, Inc., Charlottesville, Virginia 22901
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I. Introduction

FOR rigid aircraft, there are standard techniques for evaluating flight stability, generally involving the application of linear system theory. The standard approach is to consider a particular flight mode of interest (e.g., steady-wings-level flight, steady turning, steady pull-up, etc.) and evaluate the stability of an equilibrium point in this regime. Because the aerodynamic forces and moments are difficult to express analytically over a large range of possible aircraft motions, most often, equilibrium point stability is evaluated based upon a set of small-perturbation linear equations.

For a morphing aircraft, in addition to fixed planform stability, we are also interested in the transient stage of flight during which the aircraft changes shape. The particular question that we want to address is how the speed of transition affects stability. We refer to this inquiry as transition stability. The term morphing aircraft as it is used to here denotes a flight vehicle that exhibits large, controlled changes in geometry during flight for the purpose of optimal performance in a given flight regime. This definition distinguishes the present considerations from morphing concepts such as wings with conformal control surfaces, which are used as aerodynamic control inputs and do not include a substantial redistribution of mass [1,2].

The assumptions used to evaluate the stability of rigid aircraft are not valid for the morphing aircraft problem. When the mass of the aircraft is substantially redistributed, the moments of inertia and aerodynamic coefficients used for rigid analysis may change significantly. As a result, there will generally be a shift of the equilibrium conditions. Hence, we can no longer speak of asymptotic stability of an equilibrium point, but only of bounded stability. An analytical evaluation of transition stability must be based on a mathematical model of the aircraft. However, as discussed in Sec. II, a detailed dynamic description of complex changes in geometry and the resulting aerodynamic interaction is likely impractical. In subsequent discussion, we attempt to justify two important assumptions of the analysis: the inertial forces due to morphing are negligible, and the aerodynamic forces are dependent solely on the instantaneous configuration of the aircraft. Both assumptions require that the geometry changes occur at a slow rate relative to the flight speed. With these two assumptions, the flight equations can be expressed as a set of parameter-dependent ordinary differential equations, where the parameter is related to the instantaneous shape of the aircraft. The manner in which the parameter is related to the aircraft geometry is not specifically restricted in any manner. Upon linearization about an equilibrium point, the equations become linear parameter varying. Transition stability can then be evaluated by determining the limit on the rate of change of the parameter. For this analysis, the theory of slowly varying systems [3–6] is applicable; the basic theory as it applies to the morphing aircraft problem is outlined in Sec. III. Straightforward application of the theory can be algebraically cumbersome and also requires explicit analytical functions for all parameter-dependent components of the equations of motion. Numerical approximations relieve these difficulties, as demonstrated in Sec. IV with an example.

II. Problem Formulation

In agreement with the traditional notation of flight mechanics, let $v = [U \quad V \quad W]^T$ and $\omega = [P \quad Q \quad R]^T$ quantify the translational and angular velocity of a fixed-body reference frame, which is confined to rotate and translate with some rigid portion of the aircraft [7,8]. Also, let $p = [p_x \quad p_y \quad p_z]^T$ quantify the local position of a material point relative to the fixed-body frame. The matrix form (i.e., reference frame dependent) of the translational and rotational equations of a morphing aircraft can be expressed in the form [9,10] $\mathcal{D}$.

$$m\dot{v} + m\tilde{\omega}v - m\tilde{r}_C\dot{\omega} - \tilde{\omega}m\tilde{r}_C\omega - 2m\dot{\tilde{r}}_C\omega + \ddot{r}_C = F \tag{1}$$

$$m\tilde{r}_C\dot{v} + m\tilde{r}_C\tilde{\omega}v + J\dot{\omega} + \tilde{\omega}J\omega + \dot{J}\omega + \int_{\mathcal{D}} \tilde{p}\,\ddot{p}\;\mathrm{d}m = M \tag{2}$$

where m is the mass, r_C locates the center of mass relative to the fixed-body frame, F and M are the applied force and moment expressed in the body frame, $\mathcal{D}$ is the spatial volume occupied by the body, and the second moment of inertia is

$$J = -\int_V \tilde{p}\,\tilde{p}\;\mathrm{d}m$$

The ($\sim$) notation in Eqs. (1) and (2) denotes the skew symmetric representation of a single-column matrix. The equations reduce to the standard rigid body flight equations when $\dot{p} = 0$ for every material point and the center of mass is chosen as the reference origin, requiring that $r_C = 0$. For the nonrigid case, we can also let the body-fixed axis move with the center of mass, in which case, again, $r_C = 0$. However, v would then be the velocity of the center of mass rather than a fixed point of the aircraft. For sufficiently large aircraft speeds, this difference will have only a small effect. Note also that the moments of inertia may become more complicated when the reference point is in relative motion.

In the wind axis, ignoring the presence of wind gusts and cross-flow, etc., the equations of motion become

$$m\dot{v}_w + m(\tilde{\omega}_{w/b} + C_{wb}\tilde{\omega}C_{bw})v_w - mC_{wb}\tilde{r}_C\dot{\omega} - mC_{wb}\tilde{\omega}\tilde{r}_C\omega - 2mC_{wb}\dot{\tilde{r}}_C\omega + mC_{wb}\ddot{r}_C = C_{wb}F \tag{3}$$

$$m\tilde{r}_C\dot{v}_w + m\tilde{r}_C(\tilde{\omega}_{w/b} + C_{wb}\tilde{\omega}C_{bw})v_w + C_{wb}J\dot{\omega} + C_{wb}(\tilde{\omega}J + \dot{J})\omega + C_{wb}\int_V \tilde{p}\,\ddot{p}\;\mathrm{d}V = C_{wb}M \tag{4}$$

where V_T is the total velocity, and thus the wind-axis velocity is $v_w = [V_T \quad 0 \quad 0]^T$. Also, C_{wb} is an orthonormal operator that transforms vectors expressed in the fixed-body axis into the wind axis, and $\omega_{w/b}$ is the wind-axis representation of the angular velocity

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*Assistant Professor, Department of Mechanical Engineering; seigler@engr.uky.edu. Senior Member AIAA.
†Research Engineer; neal@barron-associates.com. Member AIAA.

of the wind axis with respect to the body axis. A more detailed description of these terms is provided in Appendix A.

A. General State Formulation

We now formulate a general state-space description of the dynamic equations. Suppose that the geometry of the aircraft is described by a set of k independent parameters $\xi = [\,\xi_1 \quad \cdots \quad \xi_k\,]^T$. Although possible, it is not necessary that the components of ξ are coordinates of the configuration space, such that every material point can be expressed in the form $p = p(\xi_1, \ldots, \xi_k)$. Also, assume that aerodynamic forces and moments are a function of the kinematic states, that is, the translational and angular velocity of the fixed-body reference frame, and the instantaneous geometry of the vehicle, quantified by the components of ξ. The aerodynamic forces are not considered to be functionally dependent on the derivatives of ξ, that is, the rates of change in geometry, for reasons that will be explained. Further, we assume that all terms dependent on the time derivative of p are negligible; specifically $\dot{J}$, $\dot{r}_C$, $\ddot{r}_C$, and

$$\int_V \tilde{p}\,\ddot{p}\ \mathrm{d}V$$

This implies that the momentum due to the relative motion of the morphing portion of the aircraft is insignificant compared to the overall momentum of the aircraft relative to the inertial frame. Based on these assumptions, the state equations can be expressed in the general form

$$\dot{x} = f(x, u, \xi) \tag{5}$$

where $x = [\,x_1 \quad \cdots \quad x_n\,]^T$ is a collection of the kinematic states of the system, and $u = [\,u_1 \quad \cdots \quad u_m\,]^T$ consists of the control inputs (e.g., elevator, rudder, aileron, etc.). The components of x and u will be dependent on the specific flight conditions under consideration and other assumptions, such as longitudinal or lateral flight conditions. We now seek to determine constraints on the magnitude of $\mathrm{d}\xi/\mathrm{d}t$ to ensure stability; that is, how fast can the aircraft change shape? A general solution to this question proves difficult and will depend on several factors, including the specified form of the control law u.

As indicated by Eq. (5), neither ξ nor its derivatives are considered as state variables in the dynamic formulation. Rather, it is assumed that each ξ_i is an exogenous time-dependent input that may vary over a large range. There are two reasons for this choice. First, we wish to avoid a detailed kinematic and kinetic description of the changes in geometry, in which case there would be a need to specify the forces that drive and constrain the motions. Instead, changes in geometry are considered program constraints [11], that is, kinematical constraints independent of the gross motion of the aircraft and external forcing. Practically speaking, this assumption cannot be true and is only approximately accurate when the structural control system performs with sufficient accuracy and robustness. Secondly, the complexity of the aerodynamic forces and moments requires that we make some approximation of the functions F and M, such as by linearization. However, because the geometry is not limited to small perturbations, linearization with respect to ξ and its derivatives is not valid. We are thus led to a linear parameter-varying system of equations, where ξ is the varied parameter. This is also why the aerodynamic forces and moments are not made to be dependent on derivatives of ξ, because they would then have to be included as state variables. It is also important to note that, in regard to practicality, finding the linear dependence of the aerodynamics on the rates of change in geometry is a difficult problem, either numerically or experimentally.

B. Analysis of the Longitudinal Equations

To elaborate on the previous discussion, we consider a specific example indicative of the types of changes that occur in morphing aircraft designs. In particular, we consider the longitudinal flight dynamics of the hypothetical aircraft design shown in Fig. 1. The

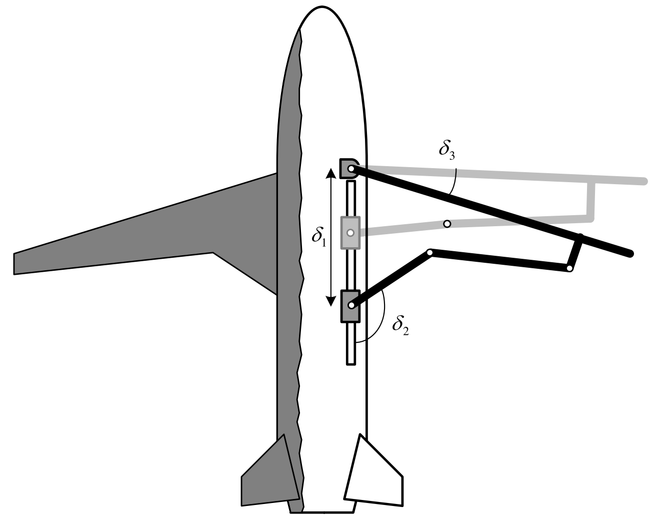

Fig. 1 Morphing aircraft example.

internal structure of the wing is controlled by three independent degrees of freedom $(\delta_1, \delta_2, \delta_3)$. Clearly, the physical realization of this design would be very complex, having various moving parts, stretching/overlapping skins, adjustable spars/ribs, etc. A complete kinematic formulation of these shape changes is likely impractical. Rather than focusing on a detailed kinematic description, we seek a small number of characteristic parameters that are representative of the change in geometry. One possibility for the current example is to let $\xi_i = \delta_i$, $i = 1, 2, 3$. However, we emphasize that the components of ξ do not necessarily have to be kinematic parameters. For instance, given some arbitrarily defined configuration A and configuration B, we may let $\xi \in [0, 1]$ denote a percentage change from A to B.

Along with the standard longitudinal assumptions $(V, P, R = 0)$, we must also assume that geometrical changes are symmetric about the local x-z plane, for otherwise lateral aerodynamic forces would become significant due to antisymmetric wing loading. Defining $r_C = [\,x_C \quad y_C \quad z_C\,]^T$, this implies that $y_C = 0$. Parameterizing rotation of the body-fixed frame by the 3-2-1 set of Euler angles (cf. Appendix A), the wind-axis equations given by Eqs. (3) and (4) reduce to

$$\begin{aligned} m\dot{V}_T + m(z_C\cos\alpha - x_C\sin\alpha)\dot{Q} &= -m\ddot{x}_C\cos\alpha - m\ddot{z}_C\sin\alpha \\ &\quad + 2mQ(\dot{x}_C\sin\alpha - \dot{z}_C\cos\alpha) + m(z_C\sin\alpha + x_C\cos\alpha)Q^2 \\ &\quad - D + T + mg\sin(\alpha - \theta) \end{aligned} \tag{6}$$

$$\begin{aligned} mV_T\dot{\alpha} - m(x_C\cos\alpha + z_C\sin\alpha)\dot{Q} &= -m\ddot{z}_C\cos\alpha + m\ddot{x}_C\sin\alpha \\ &\quad + mV_TQ + 2mQ(\dot{x}_C\cos\alpha + \dot{z}_C\sin\alpha) + m(z_C\cos\alpha \\ &\quad - x_C\sin\alpha)Q^2 - L + mg\cos(\alpha - \theta) \end{aligned} \tag{7}$$

$$\begin{aligned} mz_C\dot{V}_T - mx_CV_T\dot{\alpha} + J_y\dot{Q} &= -\dot{J}_yQ + \int_V (p_x\ddot{p}_z - p_z\ddot{p}_x)\,\mathrm{d}m \\ &\quad - mx_CV_TQ + M - mg(z_C\sin\theta + x_C\cos\theta) \end{aligned} \tag{8}$$

where θ is the pitch angle, α is the angle of attack, g is the gravitational constant, D is the drag, T is the thrust, L is the lift, M is the aerodynamic moment, and J_y is the appropriate component of the inertia tensor J. A state variable description of Eqs. (6–8) might be, for example, $x = [\,V_T \quad \alpha \quad \theta \quad Q\,]^T$ and $u = [\,\delta_e \quad \delta_T\,]^T$, the elevator deflection and thrust input, respectively. Reference errors and integral states may also be included as components of the state description.

For the current example, it is unreasonable to construct an analytical expression for the p-dependent terms of Eqs. (6–8). It is thus appropriate to consider the significance of these terms. In particular, those terms that include the time derivative of p, which represent the relative momentum (translational and angular) caused by changes in geometry, should be considered. For typical aircraft speeds, the relative momentum will be only a small fraction of the gross momentum and the aerodynamic forces. Thus, in many circumstances in which the aircraft speed is significantly greater than the relative motion due to morphing (i.e., V_T is significantly larger than $\dot{x}_c$ and $\dot{z}_c$) and the relative mass is small, it becomes reasonable to ignore those terms involving the time derivative of p. This will be considered later in the example of Sec. IV.

The most difficult part of modeling process lies in the functional description of the aerodynamic forces and moments, which are assumed to be dependent on the kinematic state of the vehicle. For instance, given the current example, we might suppose that the functional relation is of the form

$$\frac{D}{\bar{q}\,\bar{S}} = C_D = C_D(\alpha, Ma, h, \delta_e, \delta_T, \xi) \tag{9}$$

$$\frac{L}{\bar{q}\,\bar{S}} = C_L = C_L(\alpha, Ma, \delta_e, \xi) \tag{10}$$

$$\frac{M}{\bar{q}\,\bar{S}\,\bar{c}} = C_m = C_{m_0}(\alpha, Ma, h, \delta_e, \xi) + \frac{\bar{c}}{2V_T}[C_{m_q}(\xi)Q + C_{m_{\dot{\alpha}}}(\xi, \xi)\dot{\alpha}] + \frac{x_R}{\bar{c}}C_L + \Delta C_{m_T}(\delta_T, Ma, h, \xi) \tag{11}$$

where we have used the notation of Stevens and Lewis [8]. This form of the aerodynamic forces and moments is similar to the typically assumed form for a rigid aircraft, but with the effects of wing geometry included, quantified by the components of ξ. As previously discussed, a more accurate description of the aerodynamics may result when dependence on derivatives of ξ are included. However, this approach cannot be accommodated with the current analysis.

Determining functional relations for the forces and moments that are accurate over a large range of the independent variables is typically not possible. To perform an analytical study of the flight equations, the functions must be approximated in some manner. Two options of the dynamic formulation are as follows: 1) maintain the nonlinear form of Eqs. (6–8) while approximating the aerodynamic forces and moments to the first-order, that is, retaining only the linear terms of a series expansion of C_D, C_L, and C_m; or 2) a full linearization of Eqs. (6–8) about some specified operating point. We will focus on the latter option, because it will allow for a more detailed answer to the posed question. In either case, a first-order approximation of the aerodynamics cannot be expected valid over a large range of ξ, which leads to the necessity of a parameter-varying system of equations.

As previously discussed, the components of ξ are not considered dynamic states of the system, but rather exogenous time-varying inputs, independent of the gross motion of the aircraft and the external forcing. These morphing inputs are thus treated in an similar manner as the aerodynamic actuator inputs. However, unlike actuator inputs, we do not linearize with respect to ξ. A linearization with respect to these inputs would only be valid for small variations, and thus not suitable for our analysis. Our interest in modeling the dynamic equations is to analyze stability over a possibly large range of ξ.

III. Stability Analysis

Here, we outline the stability theory of slowly varying systems that will be applied to the morphing aircraft problem. The basic theory can be found in several references [3–6]. The presentation here is similar to that of Khalil [4]. The analysis is based on the system of equations previously stated by Eq. (5). We discuss two types of stability analysis. The first method is similar to control-fixed stability [7] analysis for rigid aircraft. For the morphing aircraft, however, we cannot expect that a fixed control input will maintain a fixed equilibrium point because, as the vehicle shape changes, in general, so too will the equilibrium point. We then assume that the control input is continuously varied to its equilibrium value; we term this quasi-fixed control. In the second case, termed specified control, the control input is allowed to take an arbitrary functional dependence on the state variables and the extraneous morphing parameters.

A. Quasi-Fixed Control

Assume the existence of two continuous functions, $x = x_0(\xi)$ and $u = u_0(\xi)$, such that

$$f(x_0(\xi), u_0(\xi), \xi) = 0$$

for all $\xi \in \Omega$, where Ω is some domain of interest. Letting $z = x - x_0(\xi)$, from Eq. (5), we have

$$\dot{z} = f(z + x_0(\xi), u_0(\xi), \xi) - \frac{\partial x_0}{\partial \xi}\dot{\xi} \tag{12}$$

where $z = 0$ is an equilibrium point of Eq. (12) for every frozen ξ. Linearizing the function f about $z = 0$ we obtain

$$\dot{z} = A(\xi)z + g(z, \xi) - \frac{\partial x_0}{\partial \xi}\dot{\xi} \tag{13}$$

where A is a parameter-varying matrix and g contains the higher-order terms of f. We consider the nominal dynamics of Eq. (13) to be

$$\dot{z} = A(\xi)z \tag{14}$$

Assuming that $z = 0$ is an asymptotically stable equilibrium of the nominal dynamics, we seek conditions on the remaining terms of Eq. (13) such that solutions are bounded. Suppose that A is Hurwitz for every fixed ξ, and that the elements of A and their first partial derivatives with respect to ξ are uniformly bounded. Let $P(\xi)$ be a positive-definite matrix that satisfies the Lyapunov equation

$$P(\xi)A(\xi) + A^T(\xi)P(\xi) = -Q \tag{15}$$

for every fixed ξ, where Q is a constant, positive-definite matrix, independent of ξ. The Lyapunov function $V(z, \xi) = z^T P(\xi) z$ is sufficient to prove that $z = 0$ is an asymptotically stable equilibrium of Eq. (14) for all $\xi \in \Omega$. Furthermore, $P(\xi)$ satisfies

$$c_1\|z\|_2^2 \le z^T P(\xi) z \le c_2\|z\|_2^2 \tag{16}$$

and

$$\left\|\frac{\partial}{\partial \xi_i}P(\xi)\right\|_2 \le \mu_i, \qquad i = 1, \ldots, k \tag{17}$$

for all $\xi \in \Omega$, where $\|\cdot\|_2$ denotes the 2-norm, and c_1 and c_2 are positive constants. In addition, suppose that there is an ϵ such that

$$\|\dot{\xi}\|_2 \le \epsilon < \frac{c_1}{c_2 c_3}\frac{r}{r + 2c_2 L/c_3} \tag{18}$$

where r is an arbitrary positive constant, and c_3 and L are positive constants subject to

$$\left\|\frac{\partial x_0}{\partial \xi}\right\|_2 \le L \tag{19}$$

$$c_3 = \sqrt{\sum_{i=1}^{k}\mu_i^2} \tag{20}$$

Then, for all initial conditions $\|z(0)\|_2 < r\sqrt{c_1/c_2}$, the solutions of Eq. (13) with $g(z, \xi) = 0$ are uniformly ultimately bounded by

$$b = \frac{2c_2^2 L\epsilon}{\Theta(c_1 - 2\epsilon c_2 c_3)} \tag{21}$$

where $\Theta \in (0, 1)$. A proof of these assertions is given in Khalil [4].

B. Specified Control

Again, assume the existence of $x_0(\xi)$ and $u_0(\xi)$, such that

$$f(x_0(\xi), u_0(\xi), \xi) = 0$$

for all $\xi \in \Omega$. Let $z = x - x_0(\xi)$ and $\Delta u = u - u_0(\xi)$, so that

$$\dot{z} = f(z + x_0(\xi), \Delta u + u_0(\xi), \xi) - \frac{\partial x_0}{\partial \xi}\dot{\xi} \tag{22}$$

Then, $z = 0$, $\Delta u = 0$ is an equilibrium point of Eq. (22), and linearization results in

$$\dot{z} = A(\xi)z + B(\xi)\Delta u + g(z, \xi) - \frac{\partial x_0}{\partial \xi}\dot{\xi} \tag{23}$$

As before, $g(z, \xi)$ is assumed to be insignificant. The nominal dynamics of Eq. (21) are defined as

$$\dot{z} = A(\xi)z + B(\xi)\Delta u \tag{24}$$

Suppose that Δu is of the form $\Delta u = K(\xi)z$, then the closed-loop system is

$$\dot{z} = [A(\xi) - B(\xi)K(\xi)]z := \bar{A}(\xi)z \tag{25}$$

Now, the stability analysis proceeds in an identical manner as outlined in Sec. III.A, where $A(\xi)$ in Eq. (15) is replaced by $\bar{A}(\xi)$ and so on. This is the standard framework of gain-scheduling control [12].

When Δu is of the more general form $\Delta u = \eta(z, \xi)$, one way to proceed is by the approximation

$$\Delta u \approx \frac{\partial \eta}{\partial z}(\xi)z := K(\xi)z$$

thus ignoring nonlinear terms of the control. When the nonlinear terms are sufficiently small, the stability analysis is expected reasonably accurate. When the nonlinear terms are not sufficiently small, stability analysis becomes more complex. In particular, finding a Lyapunov function $V(z, \xi)$ that proves asymptotic stability of the equilibrium point is not as simple to obtain. We do not consider this case any further here.

C. Discussion

The analysis of slowly varying systems provides a method to determine limits on the rate change of ξ in the case that $g(z, \xi) = 0$. However, we expect the results are accurate if $\|g(z, \xi)\|_2$ is sufficiently small. Determining how small can be difficult, especially in the case that $z = 0$ is not an exponentially, or at least asymptotically, stable equilibrium of Eq. (13) with $g(z, \xi) = 0$. However, $z = 0$ is only guaranteed to be a bounded equilibrium point when certain conditions are met. In application to the flight equations, the components of $g(z, \xi)$ consist of higher-order, nonlinear terms of the inertial, aerodynamic, and control forces; the relative momentum terms that are assumed negligible can also be lumped into this function if t is added as an additional independent variable. For the present analysis, we let $g(z, \xi) = 0$, with the assumption that its magnitude is small enough to be accounted for by the typically conservative results of Lyapunov stability analysis.

The value of ϵ that is derived from this analysis is not unique because the Lyapunov equation, Eq. (15), is dependent on the choice of Q. It is natural to seek the largest possible value of ϵ, and so the question is how to choose this Q accordingly. It is clear that ϵ is increased as the ratio c_2/c_1 becomes small. A common specification is $c_1 = \lambda_{\min}(P)$ and $c_2 = \lambda_{\max}(P)$, the maximum and minimum eigenvalues of $P(\xi)$ evaluated over all $\xi \in \Omega$. We then seek a Q that minimizes the ratio $\lambda_{\max}(P)/\lambda_{\min}(P)$. A procedure for finding this Q is given by Khusainov and Yun'kova [13]. Following this procedure, suppose that for every ξ we find a $Q(\xi)$ that minimizes $\lambda_{\max}[P(\xi)]/\lambda_{\min}[P(\xi)]$. Denote ξ^* as the member of Ω such that $\lambda_{\max}[P(\xi^*)]/\lambda_{\min}[P(\xi^*)] < \lambda_{\max}[P(\xi)]/\lambda_{\min}[P(\xi)]$ for all $\xi \in \Omega$. Then, setting $Q = Q(\xi^*)$ in Eq. (15), it follows that $c_1 = \lambda_{\min}[P(\xi^*)]$, and $c_2 = \lambda_{\max}[P(\xi^*)]$ satisfies the inequality.

A further increase in ϵ can be accomplished by specifying realistic bounds on z. That is, we specify a domain Γ and a $z_{\max}$ such that $\|z\|_2 \le \|z_{\max}\|_2$ for all $z \in \Gamma$. Because P is a symmetric positive-definite matrix, it follows that $z^T P z \le z_{\max}^T P z_{\max}$. The inequality of Eq. (16) is then satisfied when

$$c_2 > \frac{z_{\max}^T P(\xi) z_{\max}}{\|z_{\max}\|_2} \tag{26}$$

for all $\xi \in \Omega$.

Another important consideration for the case of specified control is the design of Δu in Eq. (24). In addition to the goal of keeping c_2/c_1 small, Eq. (18) indicates that c_3 should also be small. According to Eq. (17), this requires that the closed-loop $\bar{A}$ matrix should not vary substantially. In the extreme case, when the closed-loop system does not change at all with ξ, c_3 goes to zero and the transition rate becomes arbitrary. Thus, the control objective for transition flight should be to maintain constant closed-loop dynamics. Of course, a primary rationale for the morphing aircraft is to obtain different closed-loop properties. Hence, in practice, it may be necessary to distinguish between fixed planform control and transition control.

IV. Application Examples

To demonstrate the application of the outlined stability analysis of the previous section, we return to the longitudinal equations previously given by Eqs. (6–8). In particular, the short period pitch-attitude dynamics are analyzed. Although typically not highly relevant in modern flight control design, the pitch-attitude problem provides a simple demonstration of the stability analysis. Other flight conditions can be analyzed in a similar manner. Because specified control is the more relevant circumstance for morphing aircraft, the examples are limited to the analysis of Sec. III.B.

We consider operating conditions of steady, wings-level flight ($\theta = Q = 0$, $\alpha = \alpha_0$), in which it is common to assume that V_T is essentially constant. Thus, ignoring the V_T dynamics of Eq. (6), the pitch-attitude equations of motion are

$$m V_T \dot{\alpha} - m(x_C \cos\alpha + z_C \sin\alpha)\dot{Q} = m V_T Q + m(z_C \sin\alpha + x_C \cos\alpha)Q^2 - \bar{q}\,\bar{S}\,C_L + mg\cos(\alpha - \theta) \tag{27}$$

$$J_y \dot{Q} - \left(m x_C V_T + \frac{\bar{q}\,\bar{S}\,\bar{c}^2}{2V_T} C_{m_{\dot{\alpha}}}\right)\dot{\alpha} = -m x_C V_T Q + \bar{q}\,\bar{S}\,\bar{c}\,C_m + \frac{\bar{q}\,\bar{S}\,\bar{c}^2}{2V_T} C_{m_q} Q + \bar{q}\,\bar{S}\,x_R C_L - mg(z_C \sin\theta + x_C \cos\theta) \tag{28}$$

where the assumed form of the aerodynamic functions are

$$C_L = C_L(\alpha, \delta_e, \xi)$$

$$C_m = C_{m_0}(\alpha, \delta_e, \xi) + \frac{\bar{c}}{2V_T}[C_{m_q}(\xi)Q + C_{m_{\dot{\alpha}}}(\xi)\dot{\alpha}] + \frac{x_R}{\bar{c}} C_L(\alpha, \delta_e, \xi)$$

Here, x_R denotes distance to the point where the aerodynamic forces and moments are specified. As previously discussed, the relative momentum terms involving the time derivative of p are assumed negligible. If the reference origin of the body-fixed frame coincides with the aerodynamic reference point, then $x_R = 0$, and x_C, z_C are nonzero functions of ξ. If, alternatively, the body-fixed origin moves with the center of mass, then x_R is a nonzero function of ξ, and x_C, z_C are identically zero. Where the stability analysis can be carried out

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} (\bar q \bar S/mV_T)C_{L_\alpha} - (g/V_T)\sin\alpha_0 & (g/V_T)\sin\alpha_0 & 1 \\ 0 & 0 & 1 \\ (\bar q \bar S \bar c/J_y)C_{m_\alpha} & 0 & (\bar q \bar S \bar c^2/2V_T J_y)C_{m_q} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} (\bar q \bar S/mV_T)C_{L_{\delta_e}} \\ 0 \\ (\bar q \bar S \bar c/Jy)C_{m_{\delta_e}} \end{bmatrix} \Delta u \tag{32}$$

analytically, it is desirable to limit the number of terms that are dependent on ξ, because their partial derivatives with respect to ξ must be evaluated per Eq. (17). The moving center of mass reference frame minimizes the number of components of the dynamics that are ξ dependent. However, we do not expect that the ξ-dependent functions of Eqs. (27) and (28) can be explicitly stated, although curve-fitting discrete data sets may be a reasonable, though tedious, approach. Rather, we will evaluate stability by numerical analysis of discrete data sets, in which case, the location of the body-fixed reference origin is irrelevant in terms of complexity.

To demonstrate practical application of the outlined stability analysis, we consider the morphing aircraft design shown in Fig. 2, which is capable of independently varying span ($\delta_s \in [0, 8]$ in.), sweep ($\delta_w \in [0, 45]$ deg), and tail extension ($\delta_t \in [0, 6]$ in.). Aerodynamic derivatives were computed using the commercial software VSAERO (vortex-separation aerodynamics) over the ranges indicated in Appendix B. The nominal flight speed of the aircraft is 300 ft/s at sea level. Equilibrium values α_0 and u_0 were evaluated using standard numerical trim routines. A more detailed description of the aircraft and aerodynamic methods used is given in Appendix B. Both the aerodynamic derivatives and the inertial moments were evaluated at maximum and minimum values of δ_s, δ_w, and δ_t. Values within the maximum and minimum planform parameters are linearly interpolated. We offer no specific guidance on the number of configurations that should be evaluated, only that more accurate results are expected as the aerodynamics and inertia are evaluated at increasing number of points of Ω, in which case, interpolation occurs over smaller increments.

To keep the equations as simple as possible, the linear aerodynamic coefficients, evaluated at a fixed point of the aircraft, are resolved to the center of mass, so that $x_R = 0$. Letting $x = [\alpha \quad \theta \quad Q]^T$ and $u = \delta_e$, Eqs. (27) and (28) can be expressed in the form of Eq. (5). We assume the existence of two functions $x_0(\xi) = [\alpha_0(\xi) \quad 0 \quad 0]^T$ and $u_0(\xi)$ such that

$$\bar q \bar S C_L(\alpha_0, u_0, \xi) + mg\cos(\alpha_0) = 0 \qquad \bar q \bar S \bar c C_{m_0}(\alpha_0, u_0, \xi) = 0$$

for every fixed $\xi \in \Omega$. The assumption here is that the existence of a nominal control input is continuously varied in a feed-forward manner to the equilibrium value. We then let

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \theta \\ Q \end{bmatrix} - \begin{bmatrix} \alpha_0(\xi) \\ 0 \\ 0 \end{bmatrix}$$

so that $z = 0$ is an equilibrium point of Eq. (22), given by

$$\dot z_1 = Q - \frac{\bar q \bar S}{mV_T} C_L + \frac{g}{V_T}\cos(\alpha - \theta) - \frac{\partial \alpha_0}{\partial \xi_1}\dot\xi_1 - \frac{\partial \alpha_0}{\partial \xi_2}\dot\xi_2 - \frac{\partial \alpha_0}{\partial \xi_3}\dot\xi_3 \tag{29}$$

$$\dot z_2 = z_3 \tag{30}$$

$$\dot z_3 = \frac{\bar q \bar S \bar c}{J_y} C_{m_0} + \frac{\bar q \bar S \bar c^2}{2V_T J_y} C_{m_q} Q \tag{31}$$

where $C_{m_{\dot\alpha}}$ is negligible for this example. The nominal dynamics of Eqs. (29–31), corresponding to Eq. (24), are

The terms of Eq. (32) that are considered explicitly dependent on ξ are α_0, J_y, C_{L_α}, C_{m_α}, and C_{m_q}. However, as previously stated, the functional relations need not be explicitly defined. Even if this were accomplished, it becomes algebraically intensive to proceed analytically, even for this relatively simple example.

Open-loop pole-zero maps of Eq. (32) are shown in Fig. 3. We consider compensation of the form

$$\Delta u = K_1(\xi)z_1 + K_2(\xi)z_2 + K_3(\xi)z_3 \tag{33}$$

where K_i, ($i = 1, 2, 3$) are parameter-dependent control gains. Applying full-state feedback, the control gains are specified such that the closed-loop poles maintained at the locations $s = -0.3$, $-2 \pm i5$ for all ξ. The resulting closed-loop pole-zero maps are shown in Fig. 4. Control gains at intermediate configurations are calculated based on the corresponding fixed model. Hence, the control gains are continuously varied to maintain the closed-loop pole locations, however, the zero locations vary with ξ. Ideally, the closed-loop poles and zeros would be made identical for all configurations using eigenstructure assignment. In this case, the transition rate becomes arbitrary. However, the analysis is still useful for considering model parameter uncertainty. That is, the analysis could be conducted while allowing the coefficients of the equations of motion to vary some percentage. This would produce a set of closed-loop systems for every configuration. The collection of all configurations would then be analyzed in the manner outlined in the following examples, although with a considerable increase in the number of computations. Here, we do not conduct such extensive analysis. Rather, the purpose is to provide a simple demonstration of the procedure.

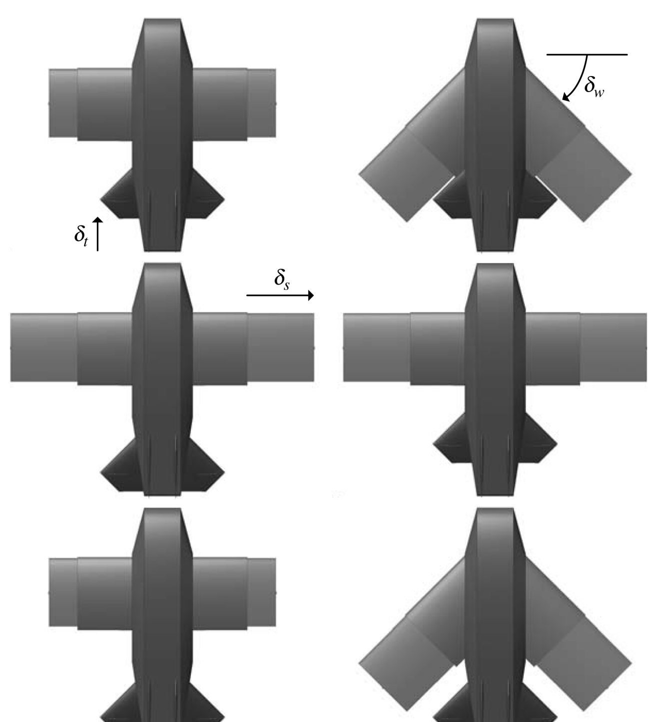

Fig. 2 Experimental adaptive planform aircraft (MORPHEUS) capable of alterations in span δ_s, tail extension δ_t, sweep δ_w, and wing twist.

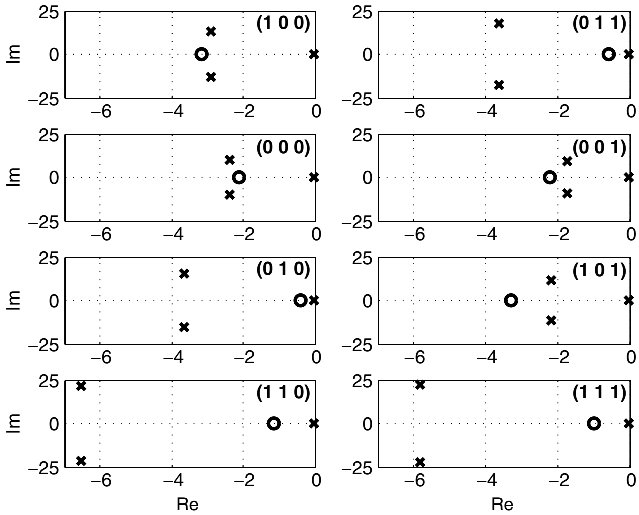

Fig. 3 Open-loop pole-zero maps for minimum and maximum configurations indicated by (δ_s δ_w δ_t) where a value of 1 indicates the maximum value and 0 the minimum.

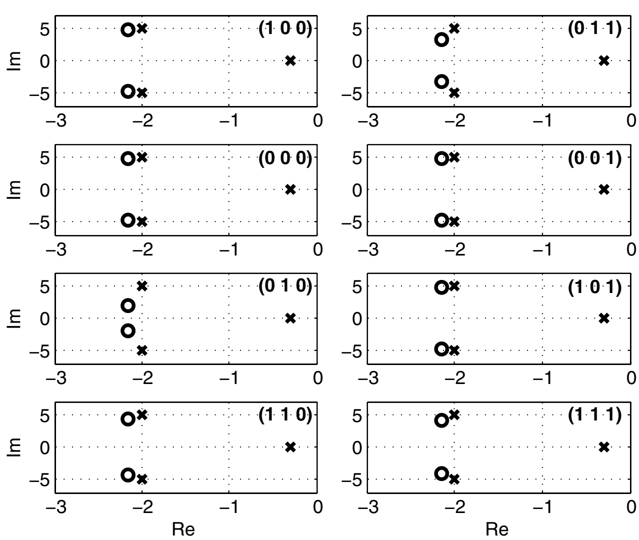

Fig. 4 Closed-loop pole-zero maps for minimum and maximum configurations indicated by (δ_s δ_w δ_t) where a value of 1 indicates the maximum value and 0 the minimum.

Example 1. Let $\xi = [\delta_s \quad \delta_w \quad \delta_t]^T$, so that the inputs are independent. The $\bar{A}(\xi)$ matrix of Eq. (25) is evaluated over the range of ξ in increments $\Delta\xi = [0.1 \text{ in.} \quad 1 \text{ deg} \quad 0.1 \text{ in.}]^T$. We find

$$\frac{\partial \alpha_0}{\partial \xi_1} \leq 1.57, \qquad \frac{\partial \alpha_0}{\partial \xi_2} \leq 0.44, \qquad \frac{\partial \alpha_0}{\partial \xi_3} \leq 0.53; \qquad \frac{\partial P}{\partial \xi_1} \leq 2.93$$

$$\frac{\partial P}{\partial \xi_2} \leq 2.83, \qquad \frac{\partial P}{\partial \xi_3} \leq 2.99; \qquad c_1 = 0.21, \qquad c_2 = 1.64$$

Suppose that the following state bounds are desired: $|\alpha| \leq 10$ deg, $|\theta| \leq 50$ deg, and $|Q| \leq 50$ deg/s. It follows that the bound of Eq. (21) is $b = 0.91$. Then, setting $L = 1.71$ and $c_3 = 5.05$, the controlled response is bounded by b for all initial conditions $\|z(0)\|_2 < 0.17$, when the transition rate is limited by $\|\mathrm{d}\xi/\mathrm{d}t\|_2 < 8.1 \times 10^{-3}$. For instance, $\delta_w < 0.4$ deg/s, $\delta_s < 0.035$ in./s, and $\delta_t < 0.035$ in./s satisfy the requirement. This transition rate covers any possible change of configuration within the specified limits. Applying the result to a specified transition between two configurations, denoted loiter ($\delta_s = 8$ in., $\delta_w = 0$ deg, $\delta_t = 0$ in.) and dash ($\delta_s = 0$ in., $\delta_w = 45$ deg, $\delta_t = 6$ in.), the minimum required time for this transition is 3.8 min. This result demonstrates the conservative nature of the analysis, even for systems that do not change substantially. Generally, more conservative results are expected from the analysis when the components of ξ are free to vary arbitrarily, because the analysis must account for the worst case. In the following example, an increase in the transition time is achieved by placing constraints on the morphing parameters.

Note that the relative momentum terms from Eqs. (6–8) were justifiably ignored, because at these transition rates the associated relative momentum terms will be very small relative to the applied forces at this flight speed. If, after the transition rate is found, it is not clear that the inertial terms are negligible, they can be calculated, or estimated, to evaluate their significance.

Example 2. Again, we consider a specified transition between loiter and dash. However, now let δ_s, δ_w, and δ_t each be continuous one-to-one functions of $\xi \in [0, 1]$. Suppose that the functional relation is given by

$$\delta_s(t) = \delta_s^* \xi(t) \qquad \delta_w(t) = \delta_w^* \xi(t) \qquad \delta_t(t) = \delta_t^* \xi(t)$$

where $(*)$ indicates the maximum value, and ξ indicates the percentage change from loiter to dash. The matrix $\bar{A}(\xi)$ was evaluated over the range of ξ in 1% increments. Following the same analysis procedure, the maximum transition rate was found to be $|\mathrm{d}\xi/\mathrm{d}t| = 0.045 \text{ s}^{-1}$, which corresponds a transition time of approximately 22 s, a significant improvement over the result of Example 1. An interesting question that arises is how the function $\xi(t)$ can be specified to minimize the stable transition rate.

V. Conclusions

An approach was presented to analyze the stability of aircraft that undergo large, controlled planform changes during flight, specifically to determine how fast an in-flight transformation may occur. Although the stability analysis outlined in this article provides an upper bound on the transition rate for morphing aircraft, it is important to note that it does not provide the least upper bound. The analysis of slowly varying systems based on Lyapunov stability theory is notoriously conservative, and the application to the morphing aircraft problem is likely to produce conservative results as well. This provides further justification, in addition to those discussed, for the linear parameter-varying description of the flight mechanics. The primary usefulness of this method is that it provides a starting point for flight testing, where a conservative estimate is typically required. In addition to the determination of less conservative estimates of transition stability, further interesting aspects of the problem include the inclusion of uncertainty analysis and optimal reconfiguration.

Appendix A: Kinematics

Rotation of the fixed-body frame $\mathcal{R}_b$ relative to the inertial frame $\mathcal{R}_r$ is parameterized by the 3-2-1 set of Euler angles $\Theta = [\phi \quad \theta \quad \psi]^T$. The matrix that transforms an $\mathcal{R}_r$ vector representation into an $\mathcal{R}_b$ vector representation is given by

$$C_{br} = \begin{bmatrix} c(\theta)c(\psi) & c(\theta)s(\psi) & -s(\theta) \\ -c(\phi)s(\psi) + s(\phi)s(\theta)c(\psi) & c(\phi)c(\psi) + s(\phi)s(\theta)s(\psi) & s(\phi)c(\theta) \\ s(\phi)s(\psi) + c(\phi)s(\theta)c(\psi) & -s(\phi)c(\psi) + c(\phi)s(\theta)s(\psi) & c(\phi)c(\theta) \end{bmatrix}$$

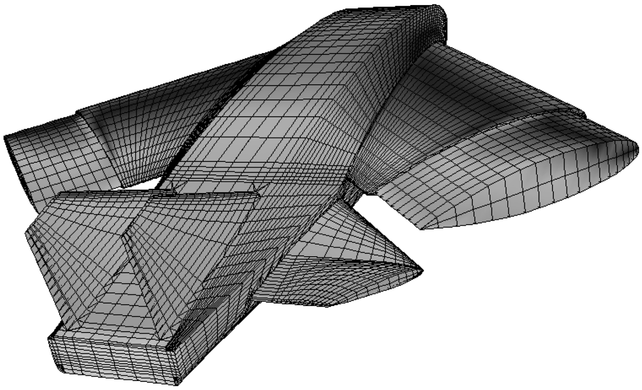

Fig. B1 VSAERO panel model.

where $C_{br}^T = C_{br}^{-1} := C_{rb}$. Let $\omega = [P \quad Q \quad R]^T$ denote the angular velocity of $\mathcal{R}_b$ with respect to $\mathcal{R}_r$, expressed in the $\mathcal{R}_b$ frame. The relationship between ω and $\dot{\Theta}$ is given by

$$\omega = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \dot{\Theta}$$

Rotation of the wind axis $\mathcal{R}_w$ relative to $\mathcal{R}_b$ is parameterized by a 3-2-1 set of Euler angles $\Phi = [\gamma \quad \alpha \quad \beta]^T$, where α is the angle of attack and β is the sideslip angle. The matrix that transforms an $\mathcal{R}_b$ vector representation into an $\mathcal{R}_w$ vector representation is given by

$$C_{wb} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

where $C_{wb}^T = C_{wb}^{-1} := C_{bw}$. Let $\omega_{w/b} := [P_w \quad Q_w \quad R_w]^T$ denote the angular velocity of $\mathcal{R}_w$ with respect to $\mathcal{R}_b$, expressed in the wind axis. The relationship between $\omega_{w/b}$ and $\dot{\Phi}$ is given by

$$\omega_{w/b} = \begin{bmatrix} 0 & -\sin\beta & 0 \\ 0 & -\cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\Phi}$$

Appendix B: Morphing Aircraft Description

The aircraft design used as an example in this study, called MORPHEUS, is an experimental adaptive planform structure previously designed and constructed at Virginia Polytechnic Institute and State University under the Defense Advanced Research Projects Agency Morphing Aircraft Structures program. There are three aerodynamic control inputs: twin rudder, symmetric and asymmetric elevator, and ± 20 deg trailing-edge wing twist. There are five pneumatic actuators: two rotational for wing twist, and three linear for span and tail extensions. Asymmetric wing sweep is controlled by two electromechanical actuators.

The aerodynamic analysis used in this work was conducted using VSAERO, which is a commercial software package that solves the classic potential flow problem for arbitrary vehicle configurations. The premise of the potential flow methodology is to use the governing equation of fluid flow and the associated boundary conditions to determine the velocity distribution about the vehicle. The aerodynamic loading is then obtained using Bernoulli's method.

Table B1 Flow conditions, control surface deflections, and planform configurations

Parameter	Range	Increment
Angle of attack, deg	−6–18	2
Sideslip, deg	−20–20	5
Control deflection, deg	−12–12	3
Span extension, in.	0–8	8
Sweep angle, deg	0–45	45
Tail contraction, in.	0–6	6

Each configuration of the vehicle is discretized into panel segments (cf. Fig. B1, Table B1) and VSAERO is used to solve the potential flow equations. VSAERO is also used to compute the damping coefficients by varying the freestream boundary conditions to account for a quasi-steady rotation and proceeding through the same analysis. The analysis is conducted for the array of flow conditions, control surface deflections, and planform configurations listed in Table B1. The control deflections include symmetric and asymmetric wing twist, symmetric and asymmetric elevator articulation, and dual rudder motions. The increments on the planform deflections indicate that data are only computed at the extremes of the planform motion which equate to eight different vehicle configurations.

The result of the analysis is a set of coefficients corresponding to the standard set of aerodynamic forces and moments. We adhere to the classical method of building up coefficients as linear combinations of nonlinear functions. Each coefficient is a combination of three nonlinear functions of angle of attack α, sideslip β, and planform deflection Δp. As an example, the moment coefficient is composed of the nominal contributions combined with the increment due to m control effectors $\delta_1, \ldots, \delta_m$ and the increment due to three quasi-steady rotations, $\omega = [P \quad Q \quad R]^T$, for example,

$$C_m = C_{m_0}(\alpha, \beta, \Delta p) + C_{m_{\delta_i}}(\alpha, \beta, \Delta p)\delta_i + C_{m_{\omega_i}}(\alpha, \beta, \Delta p)\omega_i$$

The instantaneous loading is then determined via interpolation of the underlying functions.

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